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On the polynomial mappings that belong to the set $\mathbf{H}^{\mathrm{R}}_{2,3}$.

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1. Introduction.

Proper holomorphic mappings from the 2-ball B^2 to the 3-ball B^3 that are class C^2 on $\bar{B}^2$ are classified completely ([1], [3]). They have the remarkable characteristic that they preserve affine hypersurfaces. Professor A. Dor, fixing his eyes on this property, defined the set $\mathbf{H}^{\mathrm{R}}_{n,m}$. In our previous paper [4], we examined the polynomial mappings of degree 3 that belong to $\mathbf{H}^{\mathrm{R}}_{2,3}$. The classification of the mappings belonging to $\mathbf{H}^{\mathrm{R}}_{2,3}$ is our main object, but it seems hard to find the means. For the purpose of finding the structure, we shall examine in this paper the polynomial mappings of degree 4 by calculating the determinants.

Definition 1. Let $D\subset\mathbf{C}^n$ be a domain. A holomorphic mapping f from D into $\mathbf{C}^m$ is said to be hyperplane-preserving if for any $(n-1)$-dimensional affine hyperplane V, there exists an $(m-1)$-dimensional affine hyperplane U such that $f(D\cap V)\subset U$. A holomorphic mapping $f: D\to\mathbf{C}^m$ is said to degenerate if there exists an affine hyperplane U of $\mathbf{C}^m$ such that $f(D)\subset U$.

Let f be a hyperplane-preserving holomorphic mapping from D into $\mathbf{C}^m$ which does not degenerate.

Definition 2. An affine hyperplane V of $\mathbf{C}^n$ is called a regular hyperplane of f if there exists only one affine hyperplane U of $\mathbf{C}^m$ such that $f(D\cap V)\subset U$ and $0\notin U$.

We denote by $\mathbf{H}^{\mathrm{R}}_{n,m}$ the family of all hyperplane-preserving holomorphic mappings from B^n to $\mathbf{C}^m$ that have regular hyperplanes.

Let f and g be two mappings from the domain of $\mathbf{C}^n$ into the domain of $\mathbf{C}^m$ such that $f(0)=g(0)=0$.

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Definition 3. We say that f is linearly equivalent to g if there exist two non-singular linear mappings T on $\mathbf{C}^n$ and S on $\mathbf{C}^m$ such that $f = T\circ g\circ S$.

If $f \in \mathbf{H}^{\mathrm{R}}_{n,\,m}$, then by [2] Lemma 4, there exists an open set D and only one holomorphic mapping f^* on D such that for $w=(w_1, w_2, \cdots, w_n)$,

$$\langle f\left(\frac{w}{\|w\|^2}\right),\ f^*(w)\rangle = 1 \tag{$*$}$$

where $\langle\ ,\ \rangle$ is an inner product and $\|w\|^2 = |w_1|^2 + \cdots + |w_n|^2$.

2. The case of polynomial mapping from $\mathbf{C}^2$ into $\mathbf{C}^3$ of degree 4.

Let $f \in \mathbf{H}^{\mathrm{R}}_{2,\,3}$ be a polynomial mapping of degree 4 such that $f(0)=0$. Then f can be expressed by $f(z) = P_1(z) + P_2(z) + P_3(z) + P_4(z)$, where $P_j(z)$ are given with row vectors $\mathrm{A}_{ij} \in \mathbf{C}^3$ as follows:

$$P_j(z) = \sum_{i=0}^{j} z_1^{j-i} z_2^{i} \mathrm{A}_{ji} \quad (j=1, 2, 3, 4).$$

Then by $(*)$,

$$\langle \|w\|^6 P_1(w) + \|w\|^4 P_2(w) + \|w\|^2 P_3(w) + P_4(w),\ f^*(w)\rangle = \|w\|^8. \tag{1}$$

By differentiating the equality above four times with respect to w_1 and w_2, we obtain the equalities

$$\langle \boldsymbol{\Omega}_i(w),\ f^*(w)\rangle = \binom{4}{i-1} \bar{w}_1^{5-i} \bar{w}_2^{i-1} \quad (i=1, 2, 3, 4, 5), \tag{2}$$

where vectors $\boldsymbol{\Omega}_i(w)$ are given as follows:

$$\boldsymbol{\Omega}_1(w) = \bar{w}_1^3 \mathrm{A}_{10} + \bar{w}_1^2 \mathrm{A}_{20} + \bar{w}_1 \mathrm{A}_{30} + \mathrm{A}_{40},$$

$$\boldsymbol{\Omega}_2(w) = \bar{w}_1^3 \mathrm{A}_{11} + 3\bar{w}_1^2 \bar{w}_2 \mathrm{A}_{10} + \bar{w}_1^2 \mathrm{A}_{21} + 2\bar{w}_1 \bar{w}_2 \mathrm{A}_{20} + \bar{w}_1 \mathrm{A}_{31} + \bar{w}_2 \mathrm{A}_{30} + \mathrm{A}_{41},$$

$$\boldsymbol{\Omega}_3(w) = 3\bar{w}_1^2 \bar{w}_2 \mathrm{A}_{11} + 3\bar{w}_1 \bar{w}_2^2 \mathrm{A}_{10} + \bar{w}_1^2 \mathrm{A}_{22} + 2\bar{w}_1 \bar{w}_2 \mathrm{A}_{21} + \bar{w}_2^2 \mathrm{A}_{20} + \bar{w}_1 \mathrm{A}_{32} + \bar{w}_2 \mathrm{A}_{31} + \mathrm{A}_{42},$$

$$\boldsymbol{\Omega}_4(w) = 3\bar{w}_1 \bar{w}_2^2 \mathrm{A}_{11} + \bar{w}_2^3 \mathrm{A}_{10} + 2\bar{w}_1 \bar{w}_2 \mathrm{A}_{22} + \bar{w}_2^2 \mathrm{A}_{21} + \bar{w}_1 \mathrm{A}_{33} + \bar{w}_2 \mathrm{A}_{32} + \mathrm{A}_{43},$$

$$\boldsymbol{\Omega}_5(w) = \bar{w}_2^3 \mathrm{A}_{11} + \bar{w}_2^2 \mathrm{A}_{22} + \bar{w}_2 \mathrm{A}_{33} + \mathrm{A}_{44}.$$

Now, consider the system of linear equations

$$\langle \boldsymbol{\Omega}_i(w), X\rangle = \binom{4}{i-1} \bar{w}_1^{5-i}\bar{w}_2^{i-1} \quad (i=1, 2, 3, 4, 5). \tag{3}$$

Then since $f\in\mathbf{H}^{\mathrm{R}}_{2,3}$, from the theorem of A. Dor, (3) has a unique solution $X=X(w)$.

Conversely, if (3) has a unique solution then by a simple calculation we can see that f has a regular hyperplane. Now put

$$\boldsymbol{\Phi}_1 = -\bar{w}_2\boldsymbol{\Omega}_4 + 4\bar{w}_1\boldsymbol{\Omega}_5$$

$$= \sum_{j=0}^{1} (-1)^{j+1}\bar{w}_1^j\bar{w}_2^{1-j} \{\bar{w}_2^3 \mathrm{A}_{1j} + (j+1)\bar{w}_2^2\mathrm{A}_{2,\,j+1} + b_j\bar{w}_2\mathrm{A}_{3,\,j+2} + c_j\mathrm{A}_{4,\,j+3}\},$$

where $b_0=1,\ b_1=3,\ c_0=1,\ c_1=4$,

$$\boldsymbol{\Phi}_2 = \bar{w}_2^2\boldsymbol{\Omega}_3 - 3\bar{w}_1\bar{w}_2\boldsymbol{\Omega}_4 + 6\bar{w}_1^2\boldsymbol{\Omega}_5$$

$$= \sum_{k=0}^{2} (-1)^k\bar{w}_1^k\bar{w}_2^{2-k} \left\{\bar{w}_2^2\mathrm{A}_{2k} + (k+1)\bar{w}_2\mathrm{A}_{3,\,k+1} + \frac{(k+1)(k+2)}{2}\mathrm{A}_{4,\,k+2}\right\},$$

$$\boldsymbol{\Phi}_3 = -\bar{w}_2^3\boldsymbol{\Omega}_2 + 2\bar{w}_1\bar{w}_2^2\boldsymbol{\Omega}_3 - 3\bar{w}_1^2\bar{w}_2\boldsymbol{\Omega}_4 + 4\bar{w}_1^3\boldsymbol{\Omega}_5$$

$$= \sum_{l=0}^{3} (-1)^{l+1}\bar{w}_1^l\bar{w}_2^{3-l} \{\bar{w}_2\mathrm{A}_{3l} + (l+1)\mathrm{A}_{4,\,l+1}\},$$

$$\boldsymbol{\Phi}_4 = \bar{w}_2^4\boldsymbol{\Omega}_1 - \bar{w}_1\bar{w}_2^3\boldsymbol{\Omega}_2 + \bar{w}_1^2\bar{w}_2^2\boldsymbol{\Omega}_3 - \bar{w}_1^3\bar{w}_2\boldsymbol{\Omega}_4 + \bar{w}_1^4\boldsymbol{\Omega}_5 = \sum_{m=0}^{4} (-1)^m\bar{w}_1^m\bar{w}_2^{4-m}\mathrm{A}_{4m}.$$

Then it holds that the equation (3) has a unique solution if and only if the following equation

$$\langle \boldsymbol{\Phi}_i, X(w)\rangle = 0 \quad (1\leqq i\leqq 4), \quad \langle \boldsymbol{\Omega}_5, X(w)\rangle = \bar{w}_2^4, \tag{4}$$

has a unique solution, that is if and only if

$$\mathrm{rank}\begin{pmatrix} \boldsymbol{\Phi}_1 \\ \boldsymbol{\Phi}_2 \\ \boldsymbol{\Phi}_3 \\ \boldsymbol{\Phi}_4 \end{pmatrix} = 2 \quad \text{and} \quad \mathrm{rank}\begin{pmatrix} \boldsymbol{\Phi}_1 \\ \boldsymbol{\Phi}_2 \\ \boldsymbol{\Phi}_3 \\ \boldsymbol{\Phi}_4 \\ \boldsymbol{\Omega}_5 \end{pmatrix} = 3 \tag{5}$$

for $w\in D$.

For two vectors $A=(a_1, a_2, a_3)$, $B=(b_1, b_2, b_3)\in\mathbf{C}^3$, we define $A\times B$ by

$$A\times B=\left(\begin{vmatrix}\bar{a}_2 & \bar{a}_3\\ \bar{b}_2 & \bar{b}_3\end{vmatrix}, \begin{vmatrix}\bar{a}_3 & \bar{a}_1\\ \bar{b}_3 & \bar{b}_1\end{vmatrix}, \begin{vmatrix}\bar{a}_1 & \bar{a}_2\\ \bar{b}_1 & \bar{b}_2\end{vmatrix}\right)$$

where $|\ |$ means the determinant.
Then we have

$$\Phi_3\times\Phi_4=\sum_{l=0}^{3}\sum_{m=0}^{4}(-1)^{l+m+1}w_1^{l+m}w_2^{7-l-m}\ \{w_2(A_{3l}\times A_{4m})+(l+1)(A_{4,\,l+1}\times A_{4m})\}\ . \quad (6)$$

Since $\langle\Phi_1, \Phi_3\times\Phi_4\rangle=0$, $\langle\Phi_2, \Phi_3\times\Phi_4\rangle=0$ it holds that for integers $0\leqq p\leqq 8$ and $0\leqq q\leqq 9$,

$$\sum_{j+l+m=p}\begin{vmatrix}A_{1j}\\ A_{3l}\\ A_{4m}\end{vmatrix}=0, \quad (7)$$

$$\sum_{j+l+m=p}\left\{(l+1)\begin{vmatrix}A_{1j}\\ A_{4,\,l+1}\\ A_{4m}\end{vmatrix}+(j+1)\begin{vmatrix}A_{2,\,j+1}\\ A_{3l}\\ A_{4m}\end{vmatrix}\right\}=0, \quad (8)$$

$$\sum_{j+l+m=p}\left\{(j+1)(l+1)\begin{vmatrix}A_{2,\,j+1}\\ A_{4,\,l+1}\\ A_{4m}\end{vmatrix}+b_j\begin{vmatrix}A_{3,\,j+2}\\ A_{3l}\\ A_{4m}\end{vmatrix}\right\}=0, \quad (9)$$

$$\sum_{j+l+m=p}\left\{b_j(l+1)\begin{vmatrix}A_{3,\,j+2}\\ A_{4,\,l+1}\\ A_{4m}\end{vmatrix}+c_j\begin{vmatrix}A_{4,\,j+3}\\ A_{3l}\\ A_{4m}\end{vmatrix}\right\}=0, \quad (10)$$

$$\sum_{j+l+m=p}c_j(l+1)\begin{vmatrix}A_{4,\,j+3}\\ A_{4,\,l+1}\\ A_{4m}\end{vmatrix}=0, \quad (11)$$

$$\sum_{k+l+m=q}\begin{vmatrix}A_{2k}\\ A_{3l}\\ A_{4m}\end{vmatrix}=0, \quad (12)$$

$$\sum_{k+l+m=q}\left\{(l+1)\begin{vmatrix}A_{2k}\\ A_{4,\,l+1}\\ A_{4m}\end{vmatrix}+(k+1)\begin{vmatrix}A_{3,\,k+1}\\ A_{3l}\\ A_{4m}\end{vmatrix}\right\}=0, \quad (13)$$

$$\sum_{k+l+m=q}\left\{(k+1)(l+1)\begin{vmatrix}A_{3,\,k+1}\\ A_{4,l+1}\\ A_{4m}\end{vmatrix}+\frac{(k+1)(k+2)}{2}\begin{vmatrix}A_{4,\,k+2}\\ A_{3l}\\ A_{4m}\end{vmatrix}\right\}=0, \quad (14)$$

$$\sum_{k+l+m=q}\frac{(k+1)(k+2)(l+1)}{2}\begin{vmatrix}A_{4,\,k+2}\\ A_{4,\,l+1}\\ A_{4m}\end{vmatrix}=0, \quad (15)$$

where $0\leqq j\leqq 1$, $0\leqq k\leqq 2$, $0\leqq l\leqq 3$ and $0\leqq m\leqq 4$.

From (11) and (15) it is easily seen that $\operatorname{rank}(A_{4j}) \leqq 2$.

Case 1. Let $\operatorname{rank}(A_{4j}) = 1$.

We may assume without loss of generality that $A_{40} \neq 0$. Then, from the equations $\langle \Phi_1, \Phi_2 \times \Phi_4 \rangle = 0$, $\langle \Phi_1, \Phi_3 \times \Phi_4 \rangle = 0$ and $\langle \Phi_2, \Phi_3 \times \Phi_4 \rangle = 0$ it is easily shown that $\{A_{40}, A_{30}\}, \cdots, \{A_{40}, A_{33}\}, \{A_{40}, A_{20}\}, \cdots, \{A_{40}, A_{22}\}$ are linearly dependent. Hence $\{A_{10}, A_{11}, A_{40}\}$ must be linearly independent.

Conversely if the above holds, then it is clear that rank $\begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \end{pmatrix} = 2$

and $\langle \Omega_5, \Phi_1 \times \Phi_4 \rangle = \sum_{m=0}^{4} (-1)^{m+1} \bar{w}_1^m \bar{w}_2^{11-m} \langle A_{11}, A_{10} \times A_{4m} \rangle$ does not vanish except an algebraic set. So f has a regular hyperplane. Besides, for any $u = (u_1, u_2) \in \mathbf{C}^2 - \{0\}$ and $\lambda \in \mathbf{C}$, there exist complex numbers $K_i = K_i(u, \tilde{u})$ $(1 \leqq i \leqq 5)$ such that $f(u + \lambda \tilde{u}) = f(u) + \lambda(-\bar{u}_2 A_{10} + \bar{u}_1 A_{11} + K_1 A_{40}) + (K_2 \lambda^2 + K_3 \lambda^3 + K_4 \lambda^4) A_{40}$ where $\tilde{u} = (-\bar{u}_2, \bar{u}_1)$. Therefore f is hyperplane-preserving.

Case 2. Let $\operatorname{rank}(A_{4j}) = 2$.

We may assume without loss of generality that $\{A_{43}, A_{44}\}$ are linearly independent. Note that $\Phi_3 \times \Phi_4 \neq 0$ except an algebraic set. Put $A_{42} = \lambda_1 A_{43} + \lambda_2 A_{44}$. If $\lambda_1 \neq 0$, consider the linear coordinate change S on $\mathbf{C}^2$ given by : $z_1 = u_1$, $z_2 = -\frac{\lambda_1}{3} u_1 + u_2$. Put $g(u_1, u_2) = f \circ S(u_1, u_2) = u_1 B_{10} + \cdots + u_1^4 B_{40} + \cdots + u_2^4 B_{44}$. Then it is easily seen that $\{B_{43}, B_{44}\}$ are linearly independent and $B_{42} = (\lambda_2 + \frac{2}{3}\lambda_1^2) B_{44}$. Therefore we may assume from the beginning that $A_{42} = k A_{44}$ for some $k \in \mathbf{C}$. Then from $p=6$ in (10), it is easily seen that $A_{32} \in L[A_{43}, A_{44}]$ (the linear subspace spanned by A_{43} and A_{44}). In this case $\{A_{33}, A_{43}, A_{44}\}$ must be linearly independent. Because if $A_{33} \in L[A_{43}, A_{44}]$, then $A_{30}, A_{31} \in L$ from $p=4, 5$ in (10), $A_{21}, A_{22} \in L$, from $p=6, 7$ in (9), $A_{20} \in L$ from $q=6$ in (13), and $A_{10}, A_{11} \in L$ from $p=6, 7$ in (8). Therefore f degenerates.

Next we show that $\{A_{40}, A_{43}\}$ are linearly dependent. Suppose that $\{A_{40}, A_{43}\}$ are linearly independent, then $A_{30} \in L[A_{40}, A_{43}]$ from $p=0$ in (10), and $\begin{vmatrix} A_{33} \\ A_{40} \\ A_{41} \end{vmatrix} = 0$ from $q=2$ in (14).

Then $A_{31} \in L[A_{40}, A_{43}]$ from $p=1$ in (10).

Therefore $\begin{vmatrix} A_{33} \\ A_{40} \\ A_{42} \end{vmatrix}=0$ from $p=2$ in (10). If either $\{A_{40}, A_{41}\}$ or $\{A_{40}, A_{42}\}$ is linearly independent, then $A_{33}\in L[A_{40}, A_{43}]$ and this shows that f degenerates as before. Next if $A_{41}=\mu A_{40}$ and $A_{42}=\nu A_{40}$, then $\begin{vmatrix} A_{33} \\ A_{40} \\ A_{43} \end{vmatrix}=0$ from $q=4$ in (14). Hence $A_{33}\in L[A_{40}, A_{43}]$. From these facts, we can put

$$A_{40}=k_1A_{43},\ A_{41}=k_2A_{43}+k_3A_{44},\ A_{42}=k_4A_{44}.$$

It is easily seen that $k_1=k_3$, $k_2=k_4=0$ from (10) and (14). By calculating the determinants, the row vectors A_{ij}'s are represented as follows:

$$\begin{aligned} &A_{10}=a_1A_{33}+a_2A_{43}-a_1eA_{44},\ A_{11}=bA_{33}+(a_2-be)A_{44}, \\ &A_{20}=cA_{33}+a_1A_{43}-ceA_{44},\ A_{21}=dA_{33}+bA_{43}+(a_1-ed)A_{44}, \\ &A_{22}=eA_{33}+(b-e^2)A_{44},\ A_{30}=k_1A_{33}+cA_{43}-ek_1A_{44}, \\ &A_{31}=dA_{43}+cA_{44},\ A_{32}=eA_{43}+dA_{44}, \\ &A_{40}=k_1A_{43},\ A_{41}=k_1A_{44},\ A_{42}=0, \end{aligned} \tag{16}$$

where $a_i(i=1, 2)$, b, c, d, e and k_1 are independent complex parameters. Now we may assume without loss of generality that $A_{33}=(1, 0, 0)$, $A_{43}=(0, 1, 0)$ and $A_{44}=(0, 0, 1)$. In this case we have that

$$\langle \boldsymbol{\Omega}_5,\ \boldsymbol{\Phi}_3\times\boldsymbol{\Phi}_4\rangle=-a_2k_1^2\bar{w}_2^{11}+2a_2k_1\bar{w}_1^3\bar{w}_2^8-a_2\bar{w}_1^6w_2^5. \tag{17}$$

When $a_2\neq 0$ then (17) does not vanish except an algebraic set. Moreover we can show that f is hyperplane-preserving by elementary computations. Let F be a holomorphic polynomial mapping given by (16) with $a_2\neq 0$.

The results of the computations we carried out can now be summarized as follows.

Theorem. Let f be a holomorphic polynomial mapping of degree 4.

(1). Let $\mathrm{rank}(A_{4j})=1$ and $A_{40}\neq 0$. Then $f\in\mathbf{H}_{2,3}^{\mathrm{R}}$ if and only if $A_{2i}, A_{3j}\in L[A_{40}]$ $(0\leqq i\leqq 2,\ 0\leqq j\leqq 3)$ and $\{A_{10}, A_{11}, A_{40}\}$ are linearly independent.

(2). Let $\mathrm{rank}(A_{4j})=2$. Then $f\in\mathbf{H}_{2,3}^{\mathrm{R}}$ is linearly equivalent to F.

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